Prace monograficzne z dydaktyki matematyki
WSPÓŁCZESNE PROBLEMY NAUCZANIA MATEMATYKI

Milan Hejný (Prague)

Equation in scheme oriented early education

Schema is understood as the memory structure that incorporates clusters of information relevant to comprehension. It gets embedded in a person's mind by repeated „stay" in a certain kind of environment (one's house, school, shopping centre). Scheme-oriented mathematics education is described and illustrated by means of creating scheme Equations supported by one semantic and three structural environments.

Acknowledgement:

1. The research was supported by VZ MSM 002 162 0862.
2. The author would like to thank Doc. Dr Ewa Swoboda for her valuable help with both, the lecture and improvement of the paper.

1. Introduction

The paper presents our recent results in study of a schema oriented mathematical education at a primary school. The most important former results can be found in (Hejný, Littler 2006, Hejný, Jirotková 2009, Hejný 2008, Jirotková 2007). Schema oriented mathematical education belongs to the stream of studies searching for an effective answer to the question: How to influence mathematics education to soften strategies focused on instruction and reproduction and to enhance strategies focused on understanding and creativity. We mention two of these theories in which the concept of schema plays an important role.

The **procept** theory introduced by Gray and Tall (1994) describes the duality between process and concept: „The ambiguity of notation allows the successful thinker the flexibility in thought to move between the process to carry out a mathematical task and the concept to be mentally manipulated as a part of the wider mental schema. Symbolism that inherently represents the amalgam of process/concept ambiguity we call a procept." . . . (p. 116).

The **A-P-O-S** theory introduced by Dubinsky and McDonald (1999) presupposes „ . . . that mathematical knowledge consists in an individual's tendency to deal with perceived mathematical problem situations by constructing mental actions (A), processes (P), and objects (O) and organizing them in schemas (S) to make sense of the situations and solve the problems." The last term is elucidate in the following way: „Finally, a schema for a certain mathematical concept is an individual's collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept".

The contemporary research in Poland and in the Czech Republic contributes to this stream with several remarkable results. The early mathematical education, namely the birth and the development of all fundamental pre-mathematical schemas is elaborated in a set of studies of Gruszczyk-Kolczynska (e.g. 1992 with A. Urbańska, 2006). In the same direction Swoboda (2006) using one particular proto-mathematical environment as the research tool describes two fundamental proto-schemas: shape and organisation. The process of creation and development of algebraic schemas of university students is described in the monograph Stehlíková (2004). This study gives a good insight into the link between schema and understanding. Finally the Semadeni's research in progress focused to **deep idea** and **deep intuition** analyses a set of the basic phenomena playing the fundamental role in understanding mathematics. To grasp the concept of mathematics knowledge Semadeni (e.g. 2002 and 2008) decomposed mathematical object to three issues: A typical mathematical object A has three basic features: 1) its deep idea, 2) its surface form, 3) its formal models[1] (Semadeni, 2002, p. 46).

Semadeni's deep idea is the core of knowledge with understanding. Our schema tries to describe processes of creation of this understanding from different angle.

2. Theoretical framework

The goal of the study is to describe a creation of a mathematical schema EQUATION by means of several didactic environments. The paper is linked to the papers (Hejný, Jirotková 2009 and Slezáková 2007) in which the concept of the didactic environment BUS, and environment WALK is deeply described and elaborated.

We will start with the brief explanation of the concepts generic model and schema. For the first time the English version of the theory of generic model was published in (Hejný 1988). For more details see (Hejný 2008).

2.1. Generic model

The process of gaining knowledge described in the theory of generic models

[1]Typowy twór matematyczny A ma wiec trzy zasadnicze interpretacje. Sa to: 1) jego idea gleboka, 2) jego formy powierzchniowe, 3) jego modele formalne.

is based on stages. Its cores are two mental lifts: the generalisation, which leads from concrete knowledge to generic knowledge and the abstraction, which leads from generic to abstract knowledge. The whole process can be depicted in the following scheme consisting of two consequent levels:

motivation → isolated models → *generalisation* → generic model(s)

generic model(s) → abstraction → abstract knowledge → crystallisation

Motivation. We see motivation as the tension which occurs in a person's mind as a result of the discrepancy between the existing and desired states of knowledge. The discrepancy comes from the difference between 'I do not know' and 'I need to know', or 'I cannot do that' and 'I want to be able to do that'.

Isolated models. First experiences of a new piece of knowledge come into mind gradually and have a long-term perspective. For instance, the concepts of equation, fraction, negative number, straight line, congruency or limit develop over many years at a preparatory level. The stage ends with the creation of the community of isolated models. In the future, other isolated models will come to a student's mind, but they will not influence the birth of the generic model. They will only differentiate more detail in it.

2.1. Generic model

In the scheme of the process of gaining knowledge, the generic model as the pivot between experiences and abstract knowledge plays a decisive role. It is created from the community of its isolated models and has two basic relationships to this community:

1. it denotes both the core of this community and the core of relationships between individual models and 2;
2. it is an example or representative of all its isolated models.

The first relationship denotes the construction of the generic model; the second denotes the way the model works.

Abstract knowledge. The generic model remains an object representative and does not allow for a higher level of structuring acquired knowledge. Therefore, the next step of knowledge development must be abstraction, i.e. disconnection from an object characteristic of a generic model. This shift is accompanied by a change of language and an object representative is exchanged for a symbolic representative.

Crystallisation. After its entrance into the cognitive structure, a new piece of knowledge begins to look for relationships with the existing knowledge. When it discovers disharmonies, the need arises to remove them by adapting the new knowledge to the previous knowledge and, at the same time, to change the previous knowledge to match the new knowledge.

Each new mental step, which plays a role in creating the new abstract knowledge, immediately becomes a part of the whole cognitive structure and plays a role in crystallisation. None of the pieces of knowledge which a student constructs has a final form and each is being polished, changed and broadened all the time. This permanent development of knowledge is a typical symbol of the quality of non-mechanical knowledge.

2.2. An example from the history

The theory of generic models describes process of gaining knowledge not only in ontogeny, but also in phylogeny. A nice example of generic modelling can be found in a textbook on solving equations by an Islamic mathematician Muhammad ibn Musa-al-Khwarizmi written in the 9th century; see Bell (1945, p. 101).

The following three quadratic equations are considered in the textbook:

$x^2 + 10x = 39 \qquad x^2 + 21 = 10x \qquad x^2 = 3x + 4$ (*)

Negative numbers were not known at that time, therefore, the equation

$x^2 + px + q = 0$ (for $p, q > 0$),

had no root. To solve the quadratic equation $x^2+px+q = 0$ was meaningful only if at least one of the numbers p, q was negative. There are three such cases and they are represented by equations (*).

The solution of each equation (*) is an example for the given class and the whole triad of solutions is then the generic model of the process of solution of the quadratic equation. A person who learnt how to solve equations (*) was able to solve any quadratic equation. He/she proceeded on the basis of analogy, i.e. he/she used the similarity of the process of solving of the given equation and the corresponding typical equation from (*). The generic model can therefore have the form of a direction, procedure, formula, diagram, graph, word, sign or hint.

2.3. Consequences for education

As far as understanding mathematics is concerned, teaching strategies based on problem solving and on class discussion are more successful than teaching strategies based on explanation. The set of problems posed to a class should be elaborated from the didactical point of view. It means that new knowledge appears in pupils' minds step-by-step as described in 2.1: one isolated model, then more isolated models, then the community of isolated models, and finally the generic model. This process may be very short (just only few seconds) but sometimes it may also be very long (several months or even years).

It this case it can be decomposed into four sub-stages.

1. The first concrete experience – the first isolated model appears and this is a source of new knowledge.
2. A gradual 'collecting' of more isolated models, which at this stage are still separate.
3. Some models begin to refer to each other and create a group. The feeling develops that these models are „the same" in a sense.
4. Finding out the reason for the „sameness", or even better, the correspondence between any two models. These models create a community.

The above sub-stages can be useful when we investigate how a new idea gradually develops in a student's mind. It often happens that a new sub-stage, not presented here, is identified and that while one of those presented does not appear at all. For instance, one pre-schooler understood the idea of three-digit numbers via money. She repeatedly looked at and rearranged a hundred-crown note, a twenty-crown note and a crown, and created 120 crowns and 121 crowns. Each time, as she placed money on the table, she said the corresponding number. Either she said „one hundred and twenty", or „one hundred and twenty and one, one hundred and twenty one". Through this game, she constructed the idea of a three-digit number with the help of two (not just one) models. More elaborated illustration is given in 3.2.

2.4. Schema

When someone asks you about the number of doors or carpets in your flat or house, you will probably be unable to give an immediate answer. However, in a little while you will answer the question with absolute certainty. You will imagine yourself walking from one room to another and counting the objects in question. The required pieces of information and many other data about your dwelling are embedded in your consciousness, as a part of the scheme of your flat. We use schemata to recognize not only our dwellings, but also our village, our relatives, interpersonal relationships at our workplace, etc.

Specialized literature gives various connotations of the term 'schemata'. The following quote by R. J. Gerrig (1991, p. 244 – 245) provides a rather loose definition that serves our purposes. „Theorists have coined the term schemata to refer to the memory structure that incorporate clusters of information relevant to comprehension .... A primary insight to schema theories is that we do not simply have isolated facts in memory. Information is gathered together in meaningful functional units."

Gerrig defines schema within a wide context of a person's life experience. Our definition is narrower and relates to mathematical schemata. It is a result of two requirements that specify our perceptions of mathematical schemata.

The first specification links the term 'schema' with the generic model theory and clarifies the birth of schema. Isolated clusters of information provide breeding ground for a schema and only a schema appears with the origination of the first generic model. A child may discover that the total of 2 apples and 3 apples equals 5 apples, the [sum] total of 2 and 3 dolls is 5 dolls, but s/he has not yet developed a schema for adding small numbers. This schema is only developed when the child has discovered that these calculations can be done by counting on fingers which thus become a generic model for adding small numbers.

The second specification states that we see schema as a dynamic organisation of heterogeneous elements. The word organisation emphasises the fact that it is not just a set of elements, but also a set of bonds between these elements. The adjective dynamic refers to both short- and long-term mutability of the set of elements and of the entire organisation.

There are two phenomena that serve as the spin bound of the scheme Equations:

1. the set of admissible manipulations of the equality and
2. the equality symbol „=". To this symbol we focus our attention in the following chapter.

2.5. Equality

Mathematicians interpret the symbol „=" as reflexive, symmetrical and transitive relation. It is not the way how the symbol is understood by pupils. Even if we look at the history we will find that "this symbol is relatively recent innovation in mathematics, ... up to the sixteenth century mathematicians got by without a symbol for equality. It wasn't because they had no symbols at all. They had symbols for numbers and operations - just none for equality." (Lakoff, Nunez, 2000, p. 376).

For pupils the equality $2 + 3 = 5$ is not the same as the equality $5 = 2 + 3$. The former equality is read as two plus three yields five. Here 5 is the result of a process of and „=" establishes the relationship between the process and the result. The later equality is understood by pupils as number five can be decomposed into two and three.

A couple years ago we pose two problems to primary pupils. The problems are in some sense linked to the G. Turnau's song „Ojciec ojca mego ojca". In both problems a solver has to decide whether the given equality is true or false. Problem 1 was posed to 3^{rd} graders, problem 2 to 5^{th} graders.

Problem 1. Father of my father = my grandfather. Is it true?

Problem 2. Mother of my grandmother = grandmother of my mother. Is it true?

In both cases nearly all pupils give a positive answer. However if we change both sides of this equality and ask whether „My grandfather = Father of my father" the majority of pupils start to hesitate.

What is the reason of such mistake and what is the reason of the hesitation? It is the fact that we often read the symbol „=" as „is". And this „is" is not the symmetrical relation. There is no doubt that „The father of my father is my grandfather". But my grandfather is either father of my father or father of my mother.

The same usage of the symbol „=" can be found in many written solving processes of pupils. For example one 7th grader solved the equation $7x-1 = 20$ as a following sequence of equalities

$x = 20 + 1 = 21 : 7 = 3.$

In fact the solving process and the result are correct. Just the writing is not in accordance with our notation. In pupils' understanding this line must be read: take 20, add 1, you will get 21 and then find the result of 21 : 7. It is 3 and you have the solution of the given equation.

Recently we pose more demanding problem using the incorrect application of the symbol "=" to about a two dozens of our students, future primary teachers.

Problem 3. Find the mistake in this proof of the equality 2,5 = 2,25. The proof go like this: It is 5 : 2 = 2 (1) where (1) denotes the reminder. It is also 9 : 4 = 2 (1). Since right sides of these two equalities are identical, the left sides are equal. Therefore 5 : 2 = 9 : 4, hence 2,5 = 2,25.

The students were surprised. After a while one of them said that the reminder 1 in the first equality is? and in the second equality it is? This is true. Hence the notation like „2 (1)" used in all Czech schools is dubious from the mathematical and hence also from the didactical point of view. Division with reminder is introduced in grade 3, but fractions at least one year later. Thus we need another way how to explain this paradox to pupils. After a valuable discussion students found the good explanation of the paradox. They used the metaphor: We know that „Anna is girl" and also „Eva is girl" But we can not write „girl = Eva" since this yields the nonsense „Anna = girl = Eva". Here Anna and Eva are individuals but „girl" is a set of individuals. Therefore the inscription „Anna = girl" is not appropriate but if we by tradition have to use it we should read it „Anna is one of girls" or „Anna belongs to the set of girls" rather than „Anna is a girl". Similarly the symbol „2(1)" does not denote a number but a set of numbers and „5 : 2 = 2(1)"is not an identity but the statement „5 : 2 is one of numbers of the set 2(1)". For pupils it is a too demanding explanation. So we can use a simple instruction instead: we can write „= 2(1)" but we can not put 2(1) = ".

3. Environments suitable for building the scheme Equation

Roughly speaking all environments can be divided in two categories: semantic and structural. Semantic environments are linked to the pupil's life experiences, and structural are not. Some environments are neither purely semantic not purely structural since they are in between of these.

3.1. Semantic environment Father Woodland

Father Woodland is a fairy tale figure who looks after different animals and organises tug-of-war games. The weakest animal is a mouse (M). Two mice are as strong as one cat (C). A cat and a mouse are as strong as a goose (G). A goose and a mouse are as strong as a dog (D). Other

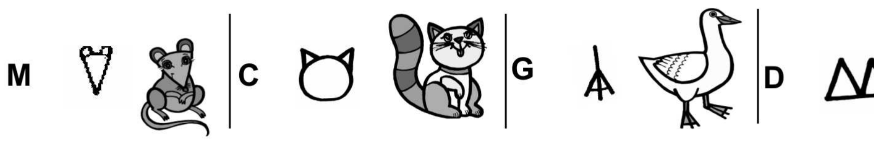

Fig. 1

animals are introduced in a similar way, too, however, they will not be considered here. Each animal is represented by both a picture and an icon (see fig. 1 – the picture was drawn by D. Raunerova).

Tug-of-war games take place on a playground which consists of two circles, one red and one blue. A group of animals go to each circle and they start pulling at a rope which lies between the circles. The task is to (a) decide which group is the stronger and (b) add some animals to the weaker group so that the two groups are equally strong. A situation in which there are two cats and one goose in the left (red) circle and a dog and a mouse in the right (blue) circle will be symbolised here as: CCG DM. Pupils got these problems in an iconic way (see fig. 2).

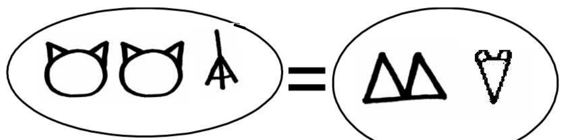

Fig. 2

Each object representing number is given to a child in three different languages: verbal and two visual ones (iconic and pictorial). Children interested in art tried to draw both - the icons and pictures. When asked to record in a graphic way {D} = {GM}, Tyna (age 7 : 4) chose pictorial language (fig. 3).

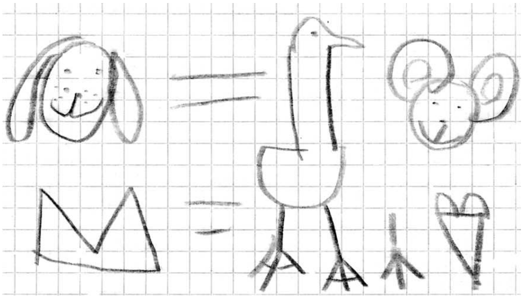

Fig. 3

This picture is very good for a seven year old. The picture of a dog is markedly different from the artist's picture of dog and the picture of mouse also lacks a noticeably triangular face. These differences points to the child's creativity and the motivational power of the context. On the other hand, children who are motivated cognitively considered drawing icons or pictures to be unnecessary. They felt that they kept them from solving problems which they found attractive. If asked to depict a situation in an iconic way, they tried to economize. From the point of view of mathematics, the second type of children seems to be more advanced. However, we feel such assessment to be one-sided if we regard the child's personality as a whole.

Note: Already in 1942 Polish mathematician Karol Borsuk created a desk game (focused on stochastic thinking) in which he used animals as the bearers of quantities (Hoffmann, 2002).

3.2. Structural environment Additive Triangle

The additive triangles frequently appeared in primary textbooks as a tool for practice addition and subtraction. They can be used also as an environment supporting understanding equations. Let us take triangle with six numbers $a,\ b,\ c,\ d,\ e,\ f$ and three identities: $a+b=d,\quad b+c=e,\quad d+e=f$.

$$\begin{array}{|c|c|c|} \hline a & b & c \\ \hline \multicolumn{3}{c}{\begin{array}{|c|c|} \hline d & e \\ \hline \end{array}} \\ \multicolumn{3}{c}{\begin{array}{|c|} \hline f \\ \hline \end{array}} \end{array}$$

Given three independent numbers we can find out the other three numbers. For example, if given $c=2$, $d=6$, $f=9$ are given, we can find $e=f-d=9-6=3$ then $b=e-c=3-2=1$ and finally $a=d-b=6-1=5$. In this example we found the unknown numbers $a,\ b,\ e$ from the given numbers $c,\ d,\ f$ directly. However, if the given numbers are $a=3$, $c=1$, $f=10$, there is no direct way (it means by one addition or one subtraction) to figure out some of the unknown numbers $b,\ d,\ e$.

In this case a pupil uses one the following two try and error methods.

a) Search for number b. A pupil will start with say $b = 2$. He/she finds $f = 8$ what is 2 below the required value 10. Thus he/she tries $b = 4$, finds $f = 12$ what is 2 above the required value 10. Now he/she puts $b = 3$ and this time he/she succeeds.

b) Decomposing number f. A pupil put $d = e = 5$. He/she find that there is no suitable b. So it is necessary to try a new decomposition of number 10. Say $d = 4$, $e = 6$. It is even worse. So try $d = 6$, $e = 4$. This time $b = 3$ is a solution.

Both try and error methods enhance pupils' ability to understand the concept of equations. After several such experiences a pupil start to look for some „magic" rule which could help him/her to find the solution quickly. Once a teacher sees strong pupils' need for some deep knowledge she can help pupils to find it using generalisation lift from isolated models to generic model. In case a) a teacher poses subsequently following set of problems to a class:

problem	I1	I2	I3	...	G1		I11	I12	I13		G2		G3		G10
$a =$	0	1	2	...	17		0	1	2		15		19		13
$c =$	0	0	0	...	0		1	1	1		1		2		18

In each of these problems numbers a and c are given and a solver has to find the rule how to figure out f if b is given. Solutions of problems I1, I2, and I3 ($f = 2b, f = 2b + 1, f = 2b + 2$) give three (or in the case of necessity) more isolated models what yields solution of G1 ($f = 2b + 17$) what is the first generic model. Similarly solutions problems I11, I12, and I13 ($f = 2b + 1, f = 2b + 1 + 1, f = 2b + 2 + 1$) give three (or in the case of necessity) more isolated models what yields solution of G2 ($f = 2b + 15 + 1$) what is the second generic model. On the same way pupils find third generic model $f = 2b + 19 + 2$ and finally the terminal generic model $f = 2b + 13 + 18$ as the solution of problem G10. In this last step a set of first three generic models ($f = 2b + 17, f = 2b + 16 + 1, f = 2b + 19 + 2$) serve as isolated models for the generic model $f = 2b + 19 + 2$.

Some pupils can find the terminal generic model very quickly and some of them can even formulate the generic model as the proto-algebraic knowledge in words: „take that b twice add the sum a + c and you have f." Some pupils need more time and more isolated models to grasp this knowledge.

When this knowledge is invented, a teacher changes the question. This time a solver has to find the rule how to figure out b if f is given. In this case $f \geqslant a+c$ and the parity of number f should be the same as the parity of $a + c$. It means that for number f we have to take numbers $a+c, a+c+2, a+c+4, a+c+6$, ... Once again pupils start with isolated modes to obtain a first, second, third, ... generic models and then, considering these to be isolated model of the higher level they find the generic model of the higher level. This result can be formulated like this „find the sum $a + c$ and subtract this number from f; the half of that result is number b".

In case b) pupils solving the similar set of problems will find that for the decomposition $f = d + e$ it is $d - e = a - c$. Further they will find the rule how to calculate number d (or e). One our fifth graders found the following rule for number d: „if $a = c$ then $d = e = f : 2$; if $a > c$ then I have to figure out the half of $a - c$ and this result I have to add to $f : 2$."

3.3. Structural environments Guess/Find my number

Frequently the tasks aimed to practice mental arithmetic. A teacher asks a question like: Grade 1: „My number is less than 9 by 2; find my number" (grade 1) or „it is twice as much as the sum 2 plus 3" (grade 2) or „if multiplied by 3 is it 20 plus 1" (grade 3) or „if half of it is multiplied by 7 you got 14" (grade 4). More demanding tasks of this kind need a paper–and–pen solving process. For example the task „if I multiply my number by 7 and add 4 I obtain 60" can hardly be solved mentally. Some external marks are necessary. Juraj's (a 5^{th} grader) record was very successful, since a boy solves this task very quickly. He put „UČ $\cdot 7 + 4 = 60$". The abbreviation UČ means „teacher's number" (in Slovak „učiteľovo číslo"). It was a common symbol for the unknown number at that class. If we replace this UČ by our symbol x, we obtain a standard equation $7x + 4 = 60$.

At first sight there is only a small difference between these two records. But in fact the difference is quite serious. A standard equation $7x + 4 = 60$ invokes in pupil's mind a set of rules for solving equations. On the other hand Juraj's record helps him to solve the task differently. When asked to explain his solution Juraj said: „I took this UČ $\cdot$ 7 as one number and since it is 4 less than 60, it is 56. But $8 \cdot 7 = 56$. Therefore teacher's number is 8.

Even more successful in recording and solving the task was Anna. She used the snake notation.

3.4. Structural environment Snake

Anna's record of teacher's task is shown in figure 4.

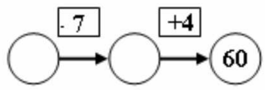

Fig. 4

Anna said: „This task is like a snake”. She drew a circle and said „here is a teacher’s number”. Then she drew the arrow with the symbol „·7” above and said „I have to multiply it by 7”. She continue „and this is the number I receive” and she drew a circle at the end of the arrow. „Now I have to subtract 4” she drew another arrow with the symbol „+4” above. „And here is the last number 60” and Anna finished the figure. Some pupils were excited and Dan said that now it is easy to solve: „start with 60, subtract 4, have 56, divide by 7, you have 8‘ excellent”.

Anna’s discovery is of fundamental importance. It is not just the new solving strategy for „Guess my number” problems. It is the discovery of the isomorphism between two environments: snakes and guess my number. The enthusiastic reaction of some pupils shows the importance of this discovery. On the level of meta–cognition it is a beginning of deep idea in Semadeni’s terminology. Two kind of knowledge are close to each other.

3.5. Conclusion

Look at figure 5. In the middle there is an equation $2x + 5 = 9$ written in traditional way. Four other representations of this equation are linked to the central rectangle.

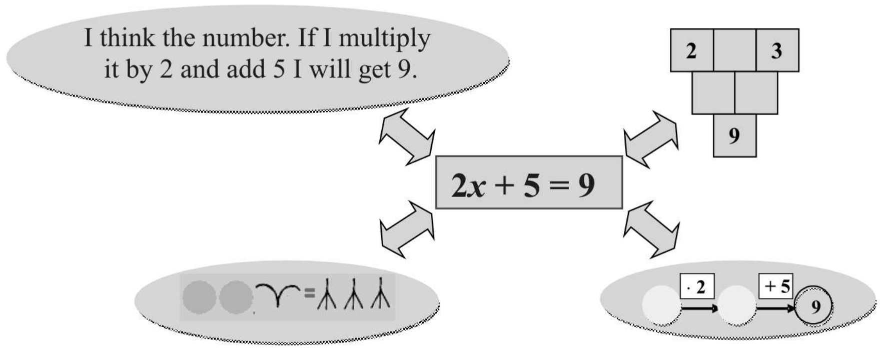

Fig. 5

A pupil who is familiar with all these environments understands the central equation deeply since he/she understands this task within a rich schema supported by four different generic models of four different environments. This schema will develop to the cognitive web which unable pupil to understand mathematics not as a set of particular definitions and instructions but as a way of thinking and understanding the world.

References

[1] Bell, E, T.: 1945, *The Development of mathematics*, McGraw-Hill, New York, London.

[2] Dubinsky E., McDonald, M.: 1999, APOS: *A Constructivist Theory of Learning in Undergraduate Mathematics Education Research*, Online www.math.kent.edu/ edd/ICMIPaper.pdf.

[3] Gerrig, R. J.: 1991, *Text comprehension*, in: The Psychology of Human Thought (eds.) R. J. Sternberg, E. E. Smith, Cambridge University Press, Cambridge, pp. 244 – 245.

[4] Gray, E., Tall, D.: 1994, *Duality, ambiguity and flexibility: A proceptual view of simple arithmetic*, Journal for Research in Mathematics Education, 25, no. 2, pp. 116 – 141.

[5] Gruszczyk-Kolczyńska, E.: 2006, *O zdolności do wychwytywania prawidłowości, czyli o zastosowaniu rytmów w procesie wspomagania rozwoju umysłowego dzieci* in: Wspomaganie dzieci z genetycznie uwarunkowanymi wadami rozwoju i ich rodzin, Polskie Towarzystwo Pedagogiczne Oddział w Poznaniu, Poznań.

[6] Gruszczyk-Kolczyńska E., Urbańska A.: 1992, *Edukacja matematyczna sześciolatków. Dziecięce liczenie. Prawidłowości pedagogiczne i psychologiczne*, Wychowanie w Przedszkolu, 5, 285 – 292

[7] Hejný, M.: 1988, *Knowledge without understanding* in Proceedings of the international symposium on research and development in mathematics education, August 3-7, Bratislava, pp. 63 – 74.

[8] Hejný, M.: 2008, *Scheme – oriented educational strategy in mathematic*, in: Supporting Independent Thinking Through Mathematica Education, Wydawnictwo Universytetu Rzeszowskiego, Rzeszów.

[9] Hejný, M., Jirotková, D.: 2009, *Didactic environment bus*, in Child and Mathematics, Visegrad Fund, Wydawnictwo Universytetu Rzeszowskiego, Rzeszów.

[10] Hejný, M., Littler, G.: 2006, *Introduction in: Creative Teaching in Mathematics* (IIATM), Pedagogicka fakulta v Praze, Praha.

[11] Hoffmann, A.: 2002, *How one can use „The Super Farmer" game in teaching mathematical modelling and problem solving*, in The Mathematics Education into the 21st Century, Palermo, September, 2002, pp. 186 – 187, http://math.unipa.it/ grim/SiHoffman.pdf.

[12] Jirotková, D.: 2007, *Budování schématu síť krychle, in Cesty zdokonalování vyučování matematice*, Jihočeská univerzita v Českých Budějovicích.

[13] Lakoff, G., Nunez, R. , E.: 2000, *Where mathematics comes from*, Basic Books, USA.

[14] Semadeni, Z.: 2008, *Deep intuition as a level in the development of the concept image*, Educational Studies in Mathematics 68, s. 1 – 17.

[15] Semadeni, Z.: 2002, *Trojaka natura matematyki: idee glebokie, formy powierzchniowe, modele formalne,* Dydaktyka matematyki 24, 41 – 92.

[16] Slezáková, J.: 2007: *Prostředí Krokování*, in Cesty zdokonalování vyučování matematice, Jihočeská univerzita v Českých Budějovicích.

[17] Stehlíková, N.: 2004, *Structural Understanding in Advanced Mathematical Thinking*, Univerzita Karlova v Praze, Pedagogická fakulta.

[18] Swoboda, E.: 2006, *Przestrzeń, regularności geometryczne i kształty w uczeniu się i nauczaniu dzieci*, Wydawnictwo Uczelniane Uniwersytetu Rzeszowskiego, Rzeszów.

Autor pracuje w Charles University in Prague,
Faculty of Education, Prague